Hilbert Spaces, Tensor Products, n-qubits.

This lecture will formalize many of the notions introduced informally in the first two lectures.

0.1 Hilbert Spaces

Consider a discrete quantum system that has k distinguishable states (e.g. a system that can be in one of k distinct energy states. The state of such a system is a unit vector in a k dimensional complex vector space $\mathscr{C}^k$. The k distinguishable states form an orthogonal basis for the vector space - say denoted by $\{|1\rangle, \ldots, |k\rangle$. Here we are using the standard inner-product over $\mathscr{C}^k$ to define orthogonality. Recall that the inner-product of two vectors $|\phi\rangle = \sum_i \alpha_i |i\rangle$ and $|\psi\rangle = \sum_i \beta_i |i\rangle$ is $\sum_i \bar{\alpha_i}\beta_i$.

Dirac's Braket Notation

We have already introduced the ket notation for vectors.

If $|v\rangle = \sum_i \alpha_i |i\rangle$ and $|w\rangle = \sum_i \beta_i |i\rangle$, then we have already observed that

$$(\vec{v}, \vec{w}) = \begin{pmatrix} \overline{\alpha_1} & \overline{\alpha_2} & \cdots & \overline{\alpha_d} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_d \end{pmatrix}.$$

We denote the row vector $(\overline{\alpha_1} \cdots \overline{\alpha_d})$ by $\langle v|$ and the inner product $(\vec{v}, \vec{w})$ by $\langle v|w\rangle$.

$\langle v|$ is a *bra*, and $|w\rangle$ is a *ket*, so $\langle v|w\rangle$ is a *braket*.

To demonstrate the utility of this notation, let $|v\rangle$ be a vector of norm 1. Define $P = |v\rangle\langle v|$. Then for any $|w\rangle$ we have $P|w\rangle = |v\rangle\langle v|w\rangle$, so P is the projection operator onto $|v\rangle$ (see diagram.) Note that $P^2 = |v\rangle\langle v|v\rangle\langle v| = P$ since $|v\rangle$ has norm 1.

More abstractly, the state of a quantum system is a unit vector in a Hilbert space. A Hilbert space is a complex vector space endowed with an inner-product and which is complete under the induced norm. The vector space axioms give us notions of span and linear independence of a set of vectors. However, to endow the vector space with geometry — the notion of angle between two vectors and the norm or length of a vector, we must define an inner-product — whose properties are listed below. The third property — completeness — is trivially satisfied for a finite dimensional system, so we will not bother to define it here.

- An *inner product* on a (complex) vector space V is a map $(\cdot,\cdot) : V \times V \to \mathscr{C}$ satisfying for each $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha, \beta \in \mathscr{C}$:

  (i) $(\vec{v}, \vec{v}) \geq 0$, and $(\vec{v}, \vec{v}) = 0$ if and only if $\vec{v} = \vec{0}$;

  (ii) $(\alpha\vec{u} + \beta\vec{v}, \vec{w}) = \alpha(\vec{u}, \vec{w}) + \beta(\vec{v}, \vec{w})$;

  (iii) $(\vec{v}, \vec{w}) = \overline{(\vec{w}, \vec{v})}$.

  An *inner product space* is a vector space together with an inner product.

- Vectors $\vec{v}, \vec{w} \in V$ are *orthogonal* if $(\vec{v}, \vec{w}) = 0$.

- A *basis* for V is a set $\{\vec{v}_1, \cdots, \vec{v}_d\}$ such that each $\vec{v} \in V$ can be written uniquely in the form $\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$. The basis is said to be *orthonormal* if $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$ for each i, j. (Here $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.)

Note that we can associate to each inner product space a canonical norm, defined by $\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}$. A *Hilbert space* is an inner product space which is complete with respect to its norm. If V is finite-dimensional (i.e. it has a finite basis), then completeness is automatically satisfied. Furthermore, there is only one Hilbert space of each dimension (up to isomorphism.)

0.2 Tensor Products

Consider two quantum systems - the first with k distinguishable (classical) states (associated Hilbert space $\mathscr{C}^k$), and the second with l distinguishable states (associated Hilbert space $\mathscr{C}^l$). What is the Hilbert space associated with the composite system? We can answer this question as follows: the number of distinguishable states of the composite system is kl — since for each distinct choice of basis (classical) state $|i\rangle$ of the first system and basis state $|j\rangle$ of the second system, we have a distinguishable state of the composite system. Thus the Hilbert space associated with the composite system is $\mathscr{C}^{kl}$.

The tensor product is a general construction that shows how to go from two vector spaces V and W of dimension k and l to a vector space $V \otimes W$ (pronounced "V tensor W") of dimension kl. Fix bases $|v_1\rangle, \ldots, |v_k\rangle$ and $|w_1\rangle, \ldots, |w_l\rangle$ for V, W respectively. Then a basis for $V \otimes W$ is given by

$$\{|v_i\rangle \otimes |w_j\rangle : 1 \leq i \leq k, 1 \leq j \leq l\},$$

so that $\dim(V \otimes W) = kl$. So a typical element of $V \otimes W$ will be of the form $\sum_{ij} \alpha_{ij}(|v_i\rangle \otimes |w_j\rangle)$. We can define an inner product on $V \otimes W$ by

$$(|v_1\rangle \otimes |w_1\rangle, |v_2\rangle \otimes |w_2\rangle) = (|v_1\rangle, |v_2\rangle) \cdot (|w_1\rangle, |w_2\rangle),$$

which extends uniquely to the whole space $V \otimes W$.

For example, consider $V = \mathscr{C}^2 \otimes \mathscr{C}^2$. V is a Hilbert space of dimension 4, so $V \cong \mathscr{C}^4$. So we can write $|00\rangle$ alternatively as $|0\rangle \otimes |0\rangle$. More generally, for n qubits we have $\mathscr{C}^2 \otimes \cdots$ (n times) $\otimes \cdots \mathscr{C}^2 \cong \mathscr{C}^{2^n}$. A typical element of this space is of the form

$$\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle.$$

A word of caution: Not all elements of $V \otimes W$ can be written as $|v\rangle \otimes |w\rangle$ for $|v\rangle \in V$, $|w\rangle \in W$. As an example, consider the Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

0.3 The Significance of Tensor Products

Classically, if we put together a subsystem that stores k bits of information with one that stores l bits of information, the total capacity of the composite system is $k + l$ bits.

From this viewpoint, the situation with quantum systems is extremely paradoxical. We need k complex numbers to describe the state of a k-level quantum system. Now consider a system that consists of a k-level subsystem and an l-level subsystem. To describe the composite system we need kl complex numbers. One might wonder where nature finds the extra storage space when we put these two subsystems together.

An extreme case of this phenomenon occurs when we consider an n qubit quantum system. The Hilbert space associated with this system is the n-fold tensor product of $\mathscr{C}^2 \equiv \mathscr{C}^{2^n}$. Thus nature must "remember" of 2^n complex numbers to keep track of the state of an n qubit system. For modest values of n of a few hundred, 2^n is larger than estimates on the number of elementary particles in the Universe.

This is the fundamental property of quantum systems that is used in quantum information processing.

Finally, note that when we actually a measure an n-qubit quantum state, we see only an n-bit string - so we can recover from the system only n, rather than 2^n, bits of information.